— Reading —

1. Read quickly Sect. 8.3.

— Exercises —

2. **Differentiable and homogeneous of degree 1 implies linear.** Let f be a differentiable function on $\mathbb{R}^n$ such that for every $h \neq 0$ and $\lambda > 0$, we have $f(\lambda h) = \lambda f(h)$. Check that $f(0) = 0$. Show that f is linear. *Hint: apply $f(0+t) = f(0) + df(0).t + ||t||\varepsilon(t)$ for $t = h$ and for $t = \lambda h$.*

3. **A function such that $\frac{\partial f}{\partial x} = 0$ but is not constant with respect to x.** Let

$$U = \mathbb{R}^2 \setminus \{(x,y) | x = 0, y \geq 0\}$$

and f the function defined on U by $f(x,y) = \begin{cases} 0, & y < 0 \\ y^2, & y \geq 0, x > 0 \\ -y^2 & y \geq 0, x < 0 \end{cases}$.

Show that f is $\mathcal{C}^1$ and $\frac{\partial f}{\partial x}(x,y) = 0$ in any point of U.

— Problems —

4. **Another expression of the tangent space of a graph.** Let ϕ be the function defined on $\mathbb{R}^3$ by $\phi(x,y,z) = x^2 + y^2 + z^2 - 1$. We note A the set $\phi^{-1}(\{0\}) \subset \mathbb{R}^3$.

   (a) We put B for the set $B = \{(x,y,z) \in A \mid z > 0\}$. Show that there exists a function f defined on some subset of $\mathbb{R}^2$ such that $B = \{(x,y,z) \mid z = f(x,y)\}$. Verify that f and ϕ are differentiable in any point.

   (b) Give an equation of the tangent space $T_{(a,b,c)}B$ of B at any point (a,b,c) in B.

   (c) Check that $T_{(a,b,c)}B$ can also be expressed as $T_{(a,b,c)}B = (a,b,c) + \ker D\phi(a,b,c)$.

   (d) If z is equal to 0, is it possible to define the tangent space of $(x,y,z) \in B$?

5. **Differentiability and eigenvectors of a self-adjoint map.** Let $<,>$ be the usual inner product on $\mathbb{R}^n$. Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a self-adjoint linear map, i.e.,

$$\forall x, y, \ < u(x), y > = < x, u(y) > .$$

   (a) Show that the map $f : x \mapsto < u(x), x >$ is differentiable and $df(x).t = 2 < u(x), t >$. *Hint: write f as the composition of $g : x \mapsto (u(x), x)$ and the bilinear map $h : (x,y) \mapsto < x, y >$.*

   (b) We put $\phi(x) = \frac{< u(x), x >}{< x, x >}$ for every $x \neq 0$. Justify that ϕ is differentiable on $\mathbb{R}^n \setminus \{0\}$. Compute $d\phi$ and show that $d\phi(a) = 0$ if and only if a is an eigenvector of u.